Erzincan Üniversitesi
Fen Bilimleri Enstitüsü Dergisi
2021, 14(3), 874 - 886
ISSN: 1307-9085, e-ISSN: 2149-4584
Araştırma Makalesi

Erzincan University
Journal of Science and Technology
2021, 14(3), 874 - 886
DOI: 10.18185/erzifbed.777540
Research Article

A Comparison of Estimation Methods for Gompertz Flexible Weibull Distribution

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Geliş / Received: 06/08/2020, Kabul / Accepted: 03/11/2021

Abstract

Gompertz flexible Weibull distribution is an extension of flexible Weibull distribution. We tackle the problem of the estimation of parameters of Gompertz flexible Weibull distribution. We discuss several methods of estimation for the Gompertz flexible Weibull distribution. The maximum likelihood estimators, least squares estimators, weighted least squares estimators, Anderson-Darling estimators, and Cramer-von Mises estimators are considered. A Monte Carlo simulation study is performed in order to compare these estimators in terms of their biases and mean square errors. According to the results of the simulation study, as the sample sizes increase the biases and mean square errors decrease in all parameter settings. Also, real data applications are presented to illustrate the usefulness of Gompertz flexible Weibull distribution and obtain estimates based on five methods of estimation.

Keywords: Gompertz flexible Weibull distribution, maximum likelihood estimators, least squares estimators, weighted least squares estimators.

Gompertz Flexible Weibull Dağılımı için Tahmin Yöntemlerinin bir Karşılaştırması

Öz

Gompertz flexible Weibull dağılımı, flexible Weibull dağılımının bir türüdür. Biz Gompertz flexible Weibull dağılımının parametrelerinin tahmini problemini ele aldık. Gompertz flexible Weibull dağılımı için birkaç tahmin yöntemi üzerinde durduk. En çok olabilirlik, en küçük kareler, ağırlıklandırılmış en küçük kareler, Anderson Darling ve Cramer-von Mises tahmin edicileri düşünülmüştür. Bu tahmin edicileri yan ve hata kareler ortalaması açısından karşılaştırabilmek için bir Monte Carlo simülasyon çalışması yapılmıştır. Simülasyon çalışmasının göre, tüm parametre durumlarında örneklem hacmi arttıkça yan ve hata kareler ortalaması azalmaktadır. Ayrıca Gompertz flexible Weibull dağılımının kullanışlılığını göstermek ve bahsedilen beş tahmin yöntemine dayalı olarak tahminleri elde etmek için gerçek veri çalışmaları sunulmuştur.

Anahtar Kelimeler: Gompertz flexible Weibull dağılımı, en çok olabilirlik tahmin edicisi, en küçük kareler tahmin edicisi, ağırlıklandırılmış en küçük kareler tahmin edicisi.

1. Introduction

Weibull distribution is a well-known lifetime distribution. It is widely used to determine the potential of wind energy. For instance, Kurban et al. (2007) examined the density of wind power and wind speed using Weibull distribution. Bebbington et al. (2007) proposed a new distribution as an alternative to Weibull distribution. Gompertz flexible Weibull distribution was introduced by Khaleel et al. (2020). It is an extension of flexible Weibull distribution suggested by Bebbington et al. (2007). We denote the Gompertz flexible Weibull distribution

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with four parameters as $GFW(\alpha,\beta,\theta,\gamma)$. The probability density function (pdf) and the cumulative density function (cdf) of $GFW(\alpha,\beta,\theta,\gamma)$ distribution are

$$f(x)=\theta\left(\alpha+\frac{\beta}{x^2}\right)\exp\left(\alpha x-\frac{\beta}{x}\right)\times\left[\exp\left(-\exp\left(\alpha x-\frac{\beta}{x}\right)\right)\right]^{-\gamma}\times\exp\left(\frac{\theta}{\gamma}\left\{1-\left[\exp\left(-\exp\left(\alpha x-\frac{\beta}{x}\right)\right)\right]^{-\gamma}\right\}\right) \quad (1)$$

and

$$F(x)=1-\exp\left(\frac{\theta}{\gamma}\left\{1-\left[\exp\left(-\exp\left(\alpha x-\frac{\beta}{x}\right)\right)\right]^{-\gamma}\right\}\right), \quad (2)$$

respectively, where α,β,θ and $\gamma>0$. Khaleel et al. (2020) studied some characteristic properties of $GFW(\alpha,\beta,\theta,\gamma)$ distribution. They illustrated the shape of the $GFW(\alpha,\beta,\theta,\gamma)$ distribution could be decreasing or unimodal via plots at different values of parameters. Similarly, it can be concluded that the shape of the failure rate could be unimodal, increasing and decreasing according to graphs of hazard function (Khaleel et al., 2020). Thus it can be said that $GFW(\alpha,\beta,\theta,\gamma)$ distribution is a very flexible distribution in data analysis. There is no study in literature since $GFW(\alpha,\beta,\theta,\gamma)$ distribution is a fairly new proposed distribution. For this reason, we consider the estimation of parameters of $GFW(\alpha,\beta,\theta,\gamma)$ distribution. The problem of parameter estimation is seen by many authors as a popular topic in recent years. There are many papers about different estimation methods for any distribution in the last decade. Asgharzadeh et al. (2011) discussed the methods of estimation for half-logistic distribution. Peng and Yan (2014) introduced a new extended Weibull distribution. They estimated the parameters of this new model using maximum likelihood estimators and Bayesian estimators in their study. Nassar et al. (2018) suggested a new distribution as an alternative Weibull distribution. They examined its properties and different estimation methods such as maximum likelihood, ordinary and weighted least squares, percentile and maximum product spacings in their paper. Tanış and Saraçoğlu (2019) compared six estimation methods for log-Kumaraswamy distribution. Ali et al. (2020a) considered a comparison of different methods of estimation for flexible Weibull distribution. Eliwa et al. (2020) focused the problem of point estimation for exponentiated odd Chen-G family of distributions. Ali et al. (2020b) provided a comparison of different methods of estimation for Logistic-Exponential distribution. Dey et al. (2020) studied the methods of point estimation for the parameters of weighted inverted Weibull distribution. Karakaya and Tanış (2020a) compared the methods of estimation for the parameter of Akash distribution. Karakaya and Tanış (2020b) discussed the methods of estimation for the parameters of Xgamma Weibull distribution. Tanış (2021) studied the estimation methods for transmuted power function distribution.

The main purpose of this paper is to estimate the parameters of the $GFW(\alpha,\beta,\theta,\gamma)$ distribution using five different methods of estimation. Therefore, maximum likelihood estimators, least squares estimators, weighted least squares estimators, Anderson-Darling estimators and Crámer–von-Mises estimators are obtained for point estimation.

The rest of this paper is organized as follows: Section 2 describes five methods of estimation (maximum likelihood method, least squares method, weighted least squares method, the method of Anderson Darling, the method of Crámer–von-Mises). An extensive Monte Carlo simulation study is performed to compare these estimators according to mean square error (MSE) criteria in Section 3. In Section 4, it is considered real data illustrations. We also provide estimates of these estimators in real data analysis. Lastly, concluding remarks are given in Section 5.

2. Methods of Estimation

In this section, five estimation methods are studied for estimating the unknown parameters of $GFW(\alpha,\beta,\theta,\gamma)$ distribution. The maximum likelihood, least-squares introduced by Swain et al. (1988), weighted least squares introduced by Swain et al. (1988), Anderson-Darling introduced by Anderson and Darling (1952) and Crámer–von-Mises methods of estimation examined by Macdonald (1971) are investigated.

Let $X_1,X_2,\ldots,X_n$ be a random sample from the $GFW(\alpha,\beta,\theta,\gamma)$ distribution. $X_{(1)}<X_{(2)}<\ldots<X_{(n)}$ symbolize the corresponding order statistics. Also, $x_{(i)}$ indicate the observed value of $X_{(i)}$ for $i=1,2,\ldots,n$. Then, the likelihood and log-likelihood function of the $GFW(\alpha,\beta,\theta,\gamma)$ distribution are given, respectively, by

$$L(\Xi)=\prod_{i=1}^{n}\theta\left(\alpha+\frac{\beta}{x_i^2}\right)\exp\left(\alpha x_i-\frac{\beta}{x_i}\right)\times\left[\exp\left(-\exp\left(\alpha x_i-\frac{\beta}{x_i}\right)\right)\right]^{-\gamma}\times\exp\left(\frac{\theta}{\gamma}\left\{1-\left[\exp\left(-\exp\left(\alpha x_i-\frac{\beta}{x_i}\right)\right)\right]^{-\gamma}\right\}\right) \quad (3)$$

and

$$\begin{aligned}\ell(\Xi)=&\,n\log(\theta)+\sum_{i=1}^{n}\log\left(\alpha+\frac{\beta}{x_i^2}\right)+\sum_{i=1}^{n}\left(\alpha x_i-\frac{\beta}{x_i}\right)-\gamma\sum_{i=1}^{n}\left(-\exp\left(\alpha x_i-\frac{\beta}{x_i}\right)\right)\\&+2\sum_{i=1}^{n}\left(\frac{\theta}{\gamma}\left\{1-\left[\exp\left(-\exp\left(\alpha x_i-\frac{\beta}{x_i}\right)\right)\right]^{-\gamma}\right\}\right)\end{aligned}, \quad (4)$$

where $\Xi=(\alpha,\beta,\theta,\gamma)$.

Then, the maximum likelihood estimator (MLE) of Ξ is given by

$$\Xi_1=\arg\max_{\Xi}\{\ell(\Xi)\}. \quad (5)$$

Let us define the following functions which are used to obtain the least-squares, weighted least squares, Anderson-Darling and Crámer–von-Mises estimates, respectively,

$$Q_{LS}(\Xi)=\sum_{i=1}^{n}\left(\left\{1-\exp\left(\frac{\theta}{\gamma}\left\{1-\left[\exp\left(-\exp\left(\alpha x_{(i)}-\frac{\beta}{x_{(i)}}\right)\right)\right]^{-\gamma}\right\}\right)\right\}-\frac{i}{n+1}\right)^2,$$

$$Q_{WLS}(\Xi)=\sum_{i=1}^{n}\frac{(n+2)(n+1)^2}{i(n-i+1)}\times\left(\left\{1-\exp\left(\frac{\theta}{\gamma}\left\{1-\left[\exp\left(-\exp\left(\alpha x_{(i)}-\frac{\beta}{x_{(i)}}\right)\right)\right]^{-\gamma}\right\}\right)\right\}-\frac{i}{n+1}\right)^2,$$

$$Q_{AD}(\Xi)=-n-\frac{1}{n}\sum_{i=1}^{n}\left[(2i-1)\times\log\left(\left\{1-\exp\left(\frac{\theta}{\gamma}\left\{1-\left[\exp\left(-\exp\left(\alpha x_{(i)}-\frac{\beta}{x_{(i)}}\right)\right)\right]^{-\gamma}\right\}\right)\right\}\right)\right]$$
$$+\frac{1}{n}\sum_{i=1}^{n}\left(\log\left(\left\{\exp\left(\frac{\theta}{\gamma}\left\{1-\left[\exp\left(-\exp\left(\alpha x_{(i)}-\frac{\beta}{x_{(i)}}\right)\right)\right]^{-\gamma}\right\}\right)\right\}\right)\right),$$

and

$$Q_{CvM}(\Xi)=\frac{1}{12n}+\sum_{i=1}^{n}\left(\left\{1-\exp\left(\frac{\theta}{\gamma}\left\{1-\left[\exp\left(-\exp\left(\alpha x_{(i)}-\frac{\beta}{x_{(i)}}\right)\right)\right]^{-\gamma}\right\}\right)\right\}-\frac{2i-1}{2n}\right)^2.$$

Then, least square estimator (LSE), weighted least square estimator (WLSE), Anderson-Darling estimator (ADE) and Crámer–von-Mises estimator (CvME) of the parameter Ξ are given, respectively by

$$\Xi_2=\arg\min_{\Xi}\{Q_{LS}(\Xi)\}, \quad (6)$$

$$\Xi_3=\arg\min_{\Xi}\{Q_{WLS}(\Xi)\}, \quad (7)$$

$$\Xi_4=\arg\min_{\Xi}\{Q_{AD}(\Xi)\}, \quad (8)$$

$$\Xi_5=\arg\min_{\Xi}\{Q_{CVM}(\Xi)\} \quad (9)$$

Five estimates given in (5)-(9) can be obtained by optim function in R with BFGS algorithm.

3. Simulation Study

In the simulation study, 5000 trials are conducted to estimate the biases and the MSEs of all estimators. Four true parameter settings are considered as $\Xi=(0.25,0.25,025,0.25)(S1)$, $(0.3,0.5,0.75,0.9)(S2)$, $(0.7,0.4,0.8,0.6)(S3)$, $(0.5,0.7,0.5,0.7)(S4)$. The sample sizes are selected as n = (50, 100, 150, 200, 250, 300, 350, 400, 450, 500). The BFGS algorithm which is available in R is used to achieve five estimates given in (5)-(9). In Tables 1-5, biases and MSEs of MLE, LSE, WLSE, ADE and CVME are reported. Tables 1-5 indicate that the biases and MSEs of five estimators are close to zero when the sample size increases as expected. Also, the MSEs of all estimators are bigger than the others for the parameter γ. It can be concluded that the MLE and WLSE generally have the smallest MSE for all parameters.

Table 1: Average biases and MSEs of MLE

		Bias				MSEs			
	n	α	β	θ	γ	α	β	θ	γ
S1	50	-0.0073	0.0178	-0.0222	0.2259	0.0060	0.0112	0.0056	0.3389
	100	-0.0058	0.0016	-0.0238	0.2080	0.0056	0.0046	0.0037	0.3139
	150	-0.0028	-0.0022	-0.0236	0.1791	0.0051	0.0027	0.0029	0.2739
	200	-0.0043	-0.0039	-0.0221	0.1707	0.0047	0.0020	0.0024	0.2386
	250	-0.0068	-0.0049	-0.0213	0.1740	0.0044	0.0016	0.0022	0.2321
	300	-0.0061	-0.0049	-0.0200	0.1604	0.0041	0.0014	0.0019	0.2057
	350	-0.0058	-0.0059	-0.0194	0.1516	0.0038	0.0012	0.0018	0.1883
	400	-0.0065	-0.0051	-0.0173	0.1408	0.0035	0.0010	0.0015	0.1640
	450	-0.0057	-0.0045	-0.0162	0.1309	0.0033	0.0009	0.0014	0.1490
	500	-0.0070	-0.0047	-0.0170	0.1360	0.0033	0.0008	0.0013	0.1488
S2	50	0.0553	-0.0034	-0.0281	0.2179	0.0286	0.0238	0.1342	1.6467
	100	0.0399	-0.0147	-0.0497	0.2056	0.0226	0.0136	0.0799	1.2926
	150	0.0323	-0.0175	-0.0548	0.2052	0.0203	0.0107	0.0666	1.1734
	200	0.0349	-0.0150	-0.0459	0.1380	0.0191	0.0079	0.0522	0.9012
	250	0.0337	-0.0159	-0.0480	0.1252	0.0184	0.0067	0.0444	0.7977
	300	0.0329	-0.0108	-0.0360	0.0878	0.0167	0.0054	0.0383	0.6602
	350	0.0340	-0.0120	-0.0400	0.0764	0.0167	0.0048	0.0343	0.6037
	400	0.0341	-0.0114	-0.0360	0.0594	0.0161	0.0042	0.0311	0.5326
	450	0.0348	-0.0079	-0.0291	0.0412	0.0158	0.0038	0.0284	0.4929
	500	0.0314	-0.0092	-0.0294	0.0421	0.0147	0.0034	0.0266	0.4461
S3	50	0.0003	-0.0120	-0.0643	0.4554	0.0933	0.0145	0.1605	1.6945
	100	-0.0183	-0.0203	-0.0948	0.4496	0.0784	0.0084	0.0878	1.4882
	150	-0.0205	-0.0197	-0.0910	0.4032	0.0719	0.0061	0.0661	1.2593
	200	-0.0224	-0.0200	-0.0904	0.3747	0.0686	0.0049	0.0559	1.0588
	250	-0.0116	-0.0173	-0.0798	0.3008	0.0617	0.0037	0.0436	0.8029
	300	-0.0098	-0.0162	-0.0772	0.2802	0.0605	0.0031	0.0382	0.7152
	350	-0.0156	-0.0156	-0.0735	0.2725	0.0564	0.0027	0.0340	0.6470
	400	-0.0047	-0.0135	-0.0661	0.2245	0.0539	0.0022	0.0278	0.5175
	450	-0.0132	-0.0137	-0.0668	0.2372	0.0526	0.0020	0.0264	0.5037
	500	-0.0098	-0.0125	-0.0645	0.2220	0.0516	0.0018	0.0239	0.4574
S4	50	0.0329	-0.0004	-0.0272	0.3198	0.0448	0.0624	0.0747	1.4437
	100	0.0071	-0.0244	-0.0459	0.3488	0.0356	0.0338	0.0405	1.2934
	150	-0.0004	-0.0268	-0.0511	0.3401	0.0327	0.0237	0.0289	1.1370
	200	-0.0070	-0.0285	-0.0522	0.3314	0.0306	0.0191	0.0243	0.9774
	250	-0.0012	-0.0270	-0.0494	0.2817	0.0288	0.0149	0.0197	0.8284
	300	0.0002	-0.0276	-0.0496	0.2633	0.0282	0.0132	0.0177	0.7595
	350	-0.0052	-0.0264	-0.0469	0.2516	0.0244	0.0113	0.0154	0.6512
	400	-0.0061	-0.0253	-0.0451	0.2412	0.0233	0.0100	0.0140	0.5972
	450	-0.0036	-0.0234	-0.0408	0.2087	0.0215	0.0086	0.0121	0.5071
	500	-0.0026	-0.0216	-0.0396	0.2015	0.0216	0.0079	0.0112	0.4865

Table 2: Average biases and MSEs of LSE

		Bias				MSEs			
	n	α	β	θ	γ	α	β	θ	γ
S1	50	-0.0016	-0.0248	-0.0225	0.2198	0.0111	0.0157	0.0059	0.2489
	100	0.0004	-0.0204	-0.0219	0.1729	0.0083	0.0069	0.0032	0.1867
	150	0.0032	-0.0179	-0.0216	0.1413	0.0070	0.0043	0.0024	0.1550
	200	0.0006	-0.0161	-0.0195	0.1326	0.0060	0.0031	0.0019	0.1340
	250	-0.0016	-0.0142	-0.0177	0.1269	0.0052	0.0025	0.0016	0.1226
	300	-0.0009	-0.0134	-0.0170	0.1175	0.0048	0.0021	0.0014	0.1116
	350	-0.0001	-0.0128	-0.0158	0.1056	0.0044	0.0018	0.0012	0.1023
	400	-0.0015	-0.0118	-0.0147	0.1038	0.0041	0.0016	0.0011	0.0969
	450	-0.0010	-0.0108	-0.0136	0.0946	0.0037	0.0014	0.0010	0.0869
	500	-0.0018	-0.0101	-0.0140	0.0971	0.0036	0.0013	0.0009	0.0866
S2	50	0.0103	-0.0398	-0.0406	0.4084	0.0383	0.0338	0.2490	1.8042
	100	0.0082	-0.0536	-0.1047	0.4817	0.0347	0.0210	0.1411	1.8508
	150	0.0101	-0.0487	-0.1023	0.4363	0.0316	0.0170	0.1173	1.6815
	200	0.0160	-0.0436	-0.0926	0.3597	0.0295	0.0138	0.0968	1.4332
	250	0.0195	-0.0416	-0.0920	0.3211	0.0283	0.0118	0.0826	1.2838
	300	0.0202	-0.0338	-0.0764	0.2731	0.0267	0.0102	0.0740	1.1365
	350	0.0210	-0.0347	-0.0812	0.2591	0.0252	0.0091	0.0671	1.0573
	400	0.0272	-0.0308	-0.0700	0.2063	0.0249	0.0082	0.0605	0.9608
	450	0.0236	-0.0289	-0.0693	0.2097	0.0240	0.0076	0.0568	0.8998
	500	0.0247	-0.0275	-0.0643	0.1867	0.0231	0.0069	0.0526	0.8460
S3	50	-0.0928	-0.0303	-0.0225	0.5587	0.1250	0.0208	0.3166	1.6901
	100	-0.0852	-0.0417	-0.1117	0.6328	0.1192	0.0128	0.1461	1.7784
	150	-0.0791	-0.0419	-0.1247	0.6236	0.1153	0.0100	0.1161	1.7638
	200	-0.0700	-0.0437	-0.1382	0.6148	0.1184	0.0086	0.1029	1.6890
	250	-0.0707	-0.0410	-0.1307	0.5667	0.1021	0.0072	0.0871	1.4876
	300	-0.0681	-0.0376	-0.1241	0.5313	0.0955	0.0062	0.0760	1.3390
	350	-0.0671	-0.0355	-0.1182	0.5058	0.0893	0.0056	0.0704	1.2525
	400	-0.0574	-0.0331	-0.1107	0.4584	0.0851	0.0050	0.0619	1.1149
	450	-0.0627	-0.0323	-0.1108	0.4618	0.0816	0.0046	0.0585	1.0766
	500	-0.0591	-0.0302	-0.1073	0.4424	0.0805	0.0042	0.0537	1.0020
S4	50	-0.0576	-0.0438	0.0094	0.4409	0.0374	0.0941	0.1619	1.0596
	100	-0.0581	-0.0668	-0.0504	0.5173	0.0377	0.0513	0.0640	1.2106
	150	-0.0567	-0.0726	-0.0709	0.5452	0.0383	0.0375	0.0466	1.2844
	200	-0.0703	-0.0829	-0.0881	0.6230	0.0368	0.0325	0.0422	1.4294
	250	-0.0694	-0.0788	-0.0863	0.5996	0.0342	0.0285	0.0378	1.3413
	300	-0.0644	-0.0775	-0.0870	0.5698	0.0328	0.0250	0.0338	1.2723
	350	-0.0613	-0.0771	-0.0876	0.5527	0.0321	0.0224	0.0310	1.2252
	400	-0.0568	-0.0698	-0.0797	0.5070	0.0300	0.0201	0.0280	1.0948
	450	-0.0589	-0.0693	-0.0787	0.4967	0.0284	0.0183	0.0258	1.0222
	500	-0.0551	-0.0647	-0.0756	0.4781	0.0289	0.0168	0.0241	0.9867

Table 3: Average biases and MSEs of WLSE

		Bias				MSEs			
	n	α	β	θ	γ	α	β	θ	γ
S1	50	0.0041	-0.0182	-0.0193	0.1706	0.0093	0.0108	0.0054	0.2237
	100	0.0047	-0.0125	-0.0165	0.1167	0.0065	0.0051	0.0027	0.1329
	150	0.0071	-0.0105	-0.0160	0.0878	0.0054	0.0030	0.0020	0.1040
	200	0.0044	-0.0094	-0.0144	0.0830	0.0046	0.0021	0.0015	0.0866
	250	0.0026	-0.0078	-0.0121	0.0735	0.0038	0.0017	0.0012	0.0717
	300	0.0034	-0.0071	-0.0115	0.0658	0.0035	0.0014	0.0010	0.0656
	350	0.0035	-0.0074	-0.0109	0.0592	0.0032	0.0012	0.0009	0.0600
	400	0.0024	-0.0064	-0.0094	0.0555	0.0029	0.0010	0.0007	0.0538
	450	0.0028	-0.0056	-0.0088	0.0503	0.0027	0.0009	0.0007	0.0492
	500	0.0020	-0.0052	-0.0092	0.0520	0.0026	0.0008	0.0006	0.0486
S2	50	0.0134	-0.0622	-0.1097	0.6533	0.0466	0.0358	0.2342	2.8505
	100	0.0101	-0.0529	-0.1138	0.5087	0.0314	0.0198	0.1278	2.0197
	150	0.0186	-0.0415	-0.0951	0.3800	0.0308	0.0144	0.0974	1.5397
	200	0.0191	-0.0335	-0.0769	0.2853	0.0243	0.0108	0.0761	1.1883
	250	0.0238	-0.0304	-0.0727	0.2334	0.0238	0.0087	0.0622	1.0097
	300	0.0247	-0.0239	-0.0593	0.1893	0.0226	0.0072	0.0540	0.8641
	350	0.0257	-0.0232	-0.0596	0.1638	0.0211	0.0062	0.0476	0.7608
	400	0.0298	-0.0202	-0.0504	0.1225	0.0204	0.0054	0.0423	0.6802
	450	0.0296	-0.0162	-0.0428	0.0996	0.0194	0.0047	0.0374	0.5977
	500	0.0278	-0.0165	-0.0416	0.0948	0.0184	0.0043	0.0347	0.5648
S3	50	-0.0565	-0.0574	-0.1396	0.8485	0.1964	0.0231	0.2785	3.1568
	100	-0.0730	-0.0486	-0.1522	0.7362	0.1268	0.0127	0.1406	2.3497
	150	-0.0526	-0.0375	-0.1256	0.5755	0.1130	0.0084	0.0977	1.7253
	200	-0.0572	-0.0353	-0.1226	0.5379	0.0992	0.0068	0.0816	1.4579
	250	-0.0527	-0.0305	-0.1078	0.4552	0.0817	0.0051	0.0637	1.1318
	300	-0.0378	-0.0266	-0.0981	0.3955	0.0824	0.0042	0.0532	0.9646
	350	-0.0378	-0.0246	-0.0918	0.3696	0.0763	0.0036	0.0474	0.8659
	400	-0.0282	-0.0220	-0.0828	0.3202	0.0716	0.0031	0.0397	0.7347
	450	-0.0377	-0.0216	-0.0832	0.3310	0.0672	0.0028	0.0371	0.7015
	500	-0.0415	-0.0197	-0.0789	0.3222	0.0611	0.0025	0.0338	0.6444
S4	50	-0.0040	-0.1028	-0.0859	0.6530	0.0956	0.0871	0.1183	2.2215
	100	-0.0328	-0.0888	-0.0920	0.6165	0.0561	0.0470	0.0591	1.8434
	150	-0.0325	-0.0736	-0.0852	0.5412	0.0479	0.0319	0.0415	1.5071
	200	-0.0462	-0.0704	-0.0839	0.5453	0.0386	0.0268	0.0360	1.3725
	250	-0.0408	-0.0612	-0.0750	0.4779	0.0347	0.0209	0.0292	1.1482
	300	-0.0335	-0.0557	-0.0702	0.4209	0.0327	0.0178	0.0251	0.9833
	350	-0.0347	-0.0533	-0.0677	0.4039	0.0297	0.0155	0.0222	0.9062
	400	-0.0311	-0.0469	-0.0603	0.3596	0.0274	0.0130	0.0191	0.7695
	450	-0.0336	-0.0459	-0.0580	0.3470	0.0244	0.0119	0.0174	0.7078
	500	-0.0305	-0.0410	-0.0537	0.3257	0.0239	0.0105	0.0156	0.6646

Table 4: Average biases and MSEs of ADE

		Bias				MSEs			
	n	α	β	θ	γ	α	β	θ	γ
S1	50	-0.0002	-0.0109	-0.0260	0.2404	0.0100	0.0098	0.0062	0.3576
	100	0.0003	-0.0123	-0.0240	0.1879	0.0076	0.0047	0.0034	0.2456
	150	0.0020	-0.0120	-0.0234	0.1593	0.0065	0.0029	0.0027	0.2094
	200	-0.0012	-0.0116	-0.0217	0.1546	0.0057	0.0021	0.0021	0.1787
	250	-0.0039	-0.0106	-0.0197	0.1493	0.0049	0.0017	0.0018	0.1625
	300	-0.0037	-0.0102	-0.0193	0.1446	0.0047	0.0015	0.0016	0.1565
	350	-0.0036	-0.0105	-0.0184	0.1360	0.0043	0.0013	0.0015	0.1451
	400	-0.0050	-0.0098	-0.0172	0.1343	0.0041	0.0011	0.0013	0.1390
	450	-0.0048	-0.0088	-0.0162	0.1271	0.0038	0.0010	0.0013	0.1292
	500	-0.0061	-0.0086	-0.0169	0.1325	0.0037	0.0009	0.0012	0.1300
S2	50	0.0073	-0.0446	-0.0845	0.6211	0.0347	0.0315	0.2174	2.7613
	100	0.0071	-0.0443	-0.1022	0.4956	0.0282	0.0183	0.1209	1.9435
	150	0.0103	-0.0376	-0.0917	0.4011	0.0261	0.0138	0.0957	1.5350
	200	0.0159	-0.0313	-0.0760	0.3019	0.0237	0.0104	0.0757	1.1993
	250	0.0172	-0.0294	-0.0740	0.2630	0.0219	0.0087	0.0629	1.0298
	300	0.0173	-0.0234	-0.0614	0.2218	0.0203	0.0073	0.0553	0.8898
	350	0.0188	-0.0232	-0.0628	0.1961	0.0194	0.0063	0.0487	0.7820
	400	0.0241	-0.0205	-0.0536	0.1512	0.0192	0.0055	0.0433	0.6989
	450	0.0235	-0.0169	-0.0471	0.1324	0.0183	0.0048	0.0390	0.6244
	500	0.0218	-0.0173	-0.0461	0.1268	0.0173	0.0044	0.0360	0.5867
S3	50	-0.1011	-0.0460	-0.1221	0.9068	0.1404	0.0201	0.2632	3.3053
	100	-0.0845	-0.0423	-0.1407	0.7392	0.1148	0.0116	0.1332	2.3064
	150	-0.0417	-0.0334	-0.1184	0.5475	0.1138	0.0080	0.0945	1.6658
	200	-0.0413	-0.0317	-0.1161	0.5015	0.1051	0.0065	0.0783	1.3951
	250	-0.0558	-0.0292	-0.1064	0.4678	0.0831	0.0050	0.0639	1.1563
	300	-0.0521	-0.0260	-0.0985	0.4273	0.0777	0.0042	0.0540	1.0021
	350	-0.0525	-0.0245	-0.0938	0.4060	0.0725	0.0037	0.0489	0.9154
	400	-0.0467	-0.0221	-0.0849	0.3616	0.0660	0.0032	0.0413	0.7808
	450	-0.0515	-0.0218	-0.0855	0.3653	0.0646	0.0028	0.0386	0.7422
	500	-0.0490	-0.0198	-0.0812	0.3464	0.0618	0.0025	0.0348	0.6801
S4	50	-0.0538	-0.0825	-0.0788	0.8128	0.0638	0.0811	0.1180	2.6592
	100	-0.0150	-0.0744	-0.0833	0.5629	0.0605	0.0444	0.0573	1.8098
	150	-0.0425	-0.0692	-0.0841	0.5834	0.0456	0.0314	0.0417	1.5879
	200	-0.0550	-0.0682	-0.0842	0.5879	0.0376	0.0270	0.0370	1.4711
	250	-0.0492	-0.0599	-0.0762	0.5174	0.0343	0.0214	0.0300	1.2208
	300	-0.0477	-0.0572	-0.0738	0.4826	0.0313	0.0186	0.0266	1.0851
	350	-0.0472	-0.0553	-0.0715	0.4614	0.0291	0.0163	0.0236	0.9996
	400	-0.0402	-0.0488	-0.0642	0.4073	0.0277	0.0139	0.0204	0.8564
	450	-0.0374	-0.0475	-0.0614	0.3789	0.0263	0.0125	0.0185	0.7744
	500	-0.0340	-0.0427	-0.0573	0.3562	0.0261	0.0111	0.0167	0.7268

Table 5: Average biases and MSEs of CVME

		Bias				MSEs			
	n	α	β	θ	γ	α	β	θ	γ
S1	50	-0.0029	-0.0016	-0.0220	0.2225	0.0105	0.0186	0.0059	0.2219
	100	-0.0009	-0.0089	-0.0221	0.1777	0.0080	0.0073	0.0032	0.1756
	150	0.0020	-0.0103	-0.0218	0.1463	0.0068	0.0044	0.0024	0.1490
	200	-0.0002	-0.0104	-0.0196	0.1353	0.0058	0.0031	0.0019	0.1270
	250	-0.0025	-0.0098	-0.0179	0.1313	0.0051	0.0025	0.0016	0.1198
	300	-0.0017	-0.0097	-0.0173	0.1214	0.0047	0.0021	0.0014	0.1090
	350	-0.0008	-0.0096	-0.0160	0.1088	0.0043	0.0018	0.0012	0.0996
	400	-0.0021	-0.0090	-0.0148	0.1066	0.0040	0.0015	0.0011	0.0942
	450	-0.0017	-0.0083	-0.0138	0.0983	0.0037	0.0014	0.0010	0.0857
	500	-0.0024	-0.0078	-0.0142	0.0998	0.0036	0.0013	0.0009	0.0851
S2	50	0.0356	-0.0029	0.0268	0.2250	0.0382	0.0337	0.2633	1.4909
	100	0.0262	-0.0316	-0.0632	0.3458	0.0351	0.0192	0.1336	1.6108
	150	0.0243	-0.0326	-0.0717	0.3300	0.0319	0.0154	0.1098	1.5033
	200	0.0288	-0.0310	-0.0680	0.2712	0.0303	0.0125	0.0903	1.3002
	250	0.0296	-0.0312	-0.0715	0.2479	0.0286	0.0107	0.0769	1.1736
	300	0.0298	-0.0247	-0.0580	0.2067	0.0269	0.0094	0.0693	1.0463
	350	0.0299	-0.0267	-0.0650	0.1994	0.0257	0.0084	0.0627	0.9814
	400	0.0351	-0.0239	-0.0558	0.1540	0.0252	0.0075	0.0568	0.8998
	450	0.0312	-0.0226	-0.0563	0.1611	0.0244	0.0070	0.0534	0.8452
	500	0.0315	-0.0218	-0.0524	0.1426	0.0234	0.0064	0.0496	0.7979
S3	50	-0.0550	-0.0038	0.0441	0.4134	0.1099	0.0211	0.3530	1.3615
	100	-0.0554	-0.0259	-0.0733	0.5147	0.1103	0.0118	0.1407	1.5007
	150	-0.0557	-0.0304	-0.0963	0.5292	0.1077	0.0090	0.1087	1.5471
	200	-0.0497	-0.0343	-0.1143	0.5316	0.1120	0.0078	0.0953	1.5011
	250	-0.0515	-0.0334	-0.1114	0.4953	0.0983	0.0065	0.0804	1.3354
	300	-0.0516	-0.0309	-0.1070	0.4685	0.0912	0.0056	0.0703	1.2109
	350	-0.0527	-0.0298	-0.1036	0.4514	0.0854	0.0051	0.0653	1.1433
	400	-0.0440	-0.0281	-0.0977	0.4093	0.0815	0.0046	0.0575	1.0227
	450	-0.0495	-0.0278	-0.0990	0.4156	0.0793	0.0042	0.0544	0.9910
	500	-0.0478	-0.0260	-0.0963	0.4008	0.0776	0.0038	0.0500	0.9252
S4	50	-0.0273	0.0083	0.0505	0.3114	0.0348	0.1012	0.1911	0.8594
	100	-0.0368	-0.0361	-0.0263	0.4095	0.0351	0.0495	0.0645	1.0115
	150	-0.0396	-0.0493	-0.0522	0.4527	0.0357	0.0352	0.0446	1.1052
	200	-0.0558	-0.0640	-0.0726	0.5436	0.0344	0.0300	0.0396	1.2789
	250	-0.0575	-0.0625	-0.0724	0.5297	0.0317	0.0263	0.0354	1.2056
	300	-0.0537	-0.0643	-0.0760	0.5109	0.0313	0.0229	0.0315	1.1608
	350	-0.0519	-0.0652	-0.0777	0.4994	0.0303	0.0205	0.0288	1.1263
	400	-0.0472	-0.0595	-0.0711	0.4575	0.0290	0.0185	0.0262	1.0081
	450	-0.0493	-0.0598	-0.0707	0.4487	0.0275	0.0169	0.0241	0.9463
	500	-0.0470	-0.0559	-0.0680	0.4355	0.0275	0.0156	0.0226	0.9184

4. Real Data Applications

In this section, five real data applications for the GFW distribution are applied. GFW distribution is fitted to the real data sets estimating the parameter using five methods of estimation given in Section 2. The MLE, LSE, WLSE, ADE and CVME of the parameters of GFW distribution are also obtained by BFGS algorithm and reported in Tables 6-8. Also, the

standard errors based on MLE of all parameters for five real data sets are given, respectively, by (0.4394, 1.9314, 33.0842, 6.2633), (0.0488, 0.9051, 0.0860, 2.6563), (0.0938, 0.8325, 1.1973, 3.1906), (0.2339, 2.7112, 0.1207, 0.8655) and (0.5331, 7.9961, 7.1098, 6.9551).
In the following we give the five real data sets;

Data Set 1 (Kundu and Raqap, 2009)

1.312, 1.314, 1.479, 1.552, 1.700, 1.803, 1.861, 1.865, 1.944, 1.958, 1.966, 1.997, 2.006, 2.021, 2.027, 2.055, 2.063, 2.098, 2.140, 2.179, 2.224, 2.240, 2.253, 2.270, 2.272, 2.274, 2.301, 2.301, 2.359, 2.382, 2.382, 2.426, 2.434, 2.435, 2.478, 2.490, 2.511, 2.514, 2.535, 2.554, 2.566, 2.570, 2.586, 2.629, 2.633, 2.642, 2.648, 2.684, 2.697, 2.726, 2.770, 2.773, 2.800, 2.809, 2.818, 2.821, 2.848, 2.880, 2.809, 2.818, 2.821, 2.848, 2.880, 2.954, 3.012, 3.067, 3.084, 3.090, 3.096, 3.128, 3.233, 3.433, 3.585, 3.585.

Data Set 2 (Nichols and Padgett, 2006)

0.39, 0.85, 1.08, 1.25, 1.47, 1.57, 1.61, 1.61, 1.69, 1.80, 1.84, 1.87, 1.89, 2.03, 2.03, 2.05, 2.12, 2.35, 2.41, 2.43, 2.48, 2.50, 2.53, 2.55, 2.55, 2.56, 2.59, 2.67, 2.73, 2.74, 2.79, 2.81, 2.82, 2.85, 2.87, 2.88, 2.93, 2.95, 2.96, 2.97, 3.09, 3.11, 3.11, 3.15, 3.15, 3.19, 3.22, 3.22, 3.27, 3.28, 3.31, 3.31, 3.33, 3.39, 3.39, 3.56, 3.60, 3.65, 3.68, 3.70, 3.75, 4.20, 4.38, 4.42, 4.70, 4.90.

Data Set 3 (Bjerkedal, 1960)

0.77, 1.74, 0.81, 1.20, 1.95, 1.20, 0.47, 1.43, 3.37, 2.20, 3.00, 3.09, 1.51, 2.10, 0.52, 1.62, 1.31, 0.32, 0.59, 0.81, 2.81, 1.87, 1.18, 1.35, 4.75, 2.48, 0.96, 1.89, 0.90, 2.05

Data Set 4 (Xu et al., 2003)

1.6, 3.5, 4.8, 5.4, 6.0, 6.5, 7.0, 7.3, 7.7, 8.0, 8.4, 2.0, 3.9, 5.0, 5.6, 6.1, 6.5, 7.1, 7.3, 7.8, 8.1, 8.4, 2.6, 4.5, 5.1, 5.8, 6.3, 6.7, 7.3, 7.7, 7.9, 8.3, 8.5, 3.0, 4.6, 5.3, 6.0, 8.7, 8.8, 9.0.

Data Set 5 (Lawless, 1982)

2.836, 3.120, 3.045, 5.169, 4.934, 4.970, 3.018, 3.770, 5.272, 3.856, 2.046

Table 6: Parameter estimation for all real data sets based on the MLE and LSE

	MLE				LSE			
Data set	α	β	θ	γ	α	β	θ	γ
1	0.3064	11.8222	36.6524	3.3635	0.2788	12.1092	42.2601	6.1689
2	0.0439	1.6871	0.0497	7.6163	0.1863	1.7785	0.0993	3.6235
3	0.0548	1.2928	0.7054	3.1252	0.0576	0.8448	0.2734	3.8642
4	0.2927	3.9180	0.1002	0.2664	0.2978	3.3609	0.1064	0.1805
5	0.2960	8.1233	1.3459	1.1444	0.1757	7.9264	2.8524	0.3722

Table 7: Parameter estimation for all real data sets based on the WLSE and ADE

	WLSE				ADE			
Data set	α	β	θ	γ	α	β	θ	γ
1	0.1863	11.2302	33.5804	16.5374	0.3661	11.8586	31.5820	2.5584
2	0.1129	1.5883	0.0559	5.5188	0.0258	2.4123	0.1320	8.4569
3	0.4331	0.7626	0.2042	4.3505	0.0389	0.9493	0.3127	4.1959
4	0.3114	2.5088	0.0796	0.1636	0.2649	3.6030	0.1129	0.3268
5	0.1797	7.8625	2.5029	0.9826	0.1829	10.0765	5.1078	0.0233

Table 8: Parameter estimation for all real data sets based on the CVME

	CVME			
Data set	α	β	θ	γ
1	0.3014	12.2587	42.2299	6.2834
2	0.1815	2.0880	0.1296	3.8467
3	0.0619	1.0264	0.39911	3.6164
4	0.3007	3.9545	0.1159	0.1874
5	0.2428	8.4646	2.6545	0.0595

5. Concluding Remark

In this paper, GFW distribution introduced by Khaleel et al. (2020) is examined in views of some point estimations. Five estimators are studied to estimate the four parameters of GFW distribution. A new extension is ensured for the estimation of the parameters for GFW distribution. Monte Carlo simulations are conducted for different parameter values and different sample sizes. It is concluded that as sizes of samples increases, the biases and especially MSEs of all estimators decreases and close to zero. In conclusion, we recommend the MLE and WLSE to estimate the parameters of GFW distribution due to they generally have smallest MSE according to simulation results. The parameter estimates of GFW distribution are obtained using five different estimation methods for five practical data sets. In future studies, the other estimation methods such as percentile, Bayes, moments methods, etc. can be discussed for point estimation of parameters of the GFW distribution.

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